Series/Sequences and Mathematical Induction Summary

1. Summation

(a) Anatomy of the sigma notation:

\[ \sum_{r=a}^{n} 5r + 2^{r} = [5(a) + 2^{a}] + [5(a + 1) + 2^{a+1}] + [5(a + 2) + 2^{a+2}] + \ldots + [5(n) + 2^{n}] \]

\[ r = a \quad r = a + 1 \quad r = a + 2 \quad r = n \]

\( r \) is the variable; it changes value from \( a \) (for the first term of the series) to \( n \) (for the last term of the series). In this particular context both \( a \) and \( n \) are fixed integer constants.

Total number of terms = \( n - a + 1 \)

Note: \( r \) need not always necessarily be assigned as the variable, so be mindful of the representations given in the question and interpret the series structure expansion correctly.

Consider the immediate 3 examples given below where shuffling in terms of naming the components have been made:

(i) \[ \sum_{a=n}^{r} 5a + 2^{a} = [5(n) + 2^{n}] + [5(n + 1) + 2^{n+1}] + [5(n + 2) + 2^{n+2}] + \ldots + [5(r) + 2^{r}] \]

(ii) \[ \sum_{a=n}^{r} 5n + 2^{n} = [5(n) + 2^{n}] + [5(n + 1) + 2^{n}] + [5(n + 2) + 2^{n}] + \ldots + [5(n) + 2^{n}] \]

\( r - n + 1 \) terms

(iii) \[ \sum_{n=a}^{r} 5n + 2^{n} = [5(a) + 2^{a}] + [5(a + 1) + 2^{a+1}] + [5(a + 2) + 2^{a+2}] + \ldots + [5(r) + 2^{r}] \]

(b) Formulas of popular summation series:

\[ \sum_{r=1}^{n} r = \frac{n}{2} (n + 1) \quad \sum_{r=1}^{n} r^2 = \frac{n}{6} (n + 1)(2n + 1) \quad \sum_{r=1}^{n} r^3 = \frac{n^2}{4} (n + 1)^2 \left[ \left( \sum_{r=1}^{n} r \right)^2 \right] \]

\[ \sum_{r=1}^{n} \ln r = \ln 1 + \ln 2 + \ln 3 + \ldots + \ln n = \ln (1 \times 2 \times 3 \times \ldots \ldots \times n) = \ln(n!) \]

Note: \( \sum_{r=2}^{n} \ln r \) also = \( \ln(n!) \) since \( \ln 1 = 0 \)
\[
\sum_{r=1}^{n} a^r = \frac{a(1-a^n)}{1-a} = \frac{a^n-1}{a-1}
\]

Applying these formulas in a strictly direct manner requires the lower bound to start from the value of 1. (This is with the exception of the natural logarithm series) Hence, adjustments must be made when this requirement is not met.

Example: \[
\sum_{r=1}^{2n} r^2 = \sum_{r=1}^{2n} r^2 - \sum_{r=1}^{n-1} r^2 = \frac{2n}{6} (2n+1)(4n+1) - \frac{n-1}{6} [(n-1)+1][2(n-1)+1] \\
= \frac{n}{3} (2n+1)(4n+1) - \frac{n-1}{6} (n)(2n-1)
\]

The condensed sigma notation may also be equivalent to certain Arithmetic Progressions and Geometric Progressions.

**AP:** \[
\sum_{r=a}^{b} f(r) \text{ where } f(r) \text{ must be a linear function in } r.
\]

Example: \[
\sum_{r=5}^{n} 3r - 1 = 14 + 17 + 20 + 23 + \ldots + (3n-1) \text{ is an AP with first term 14, common difference of 3 and a total of } n-5+1=n-4 \text{ terms.}
\]

Hence, \[
\sum_{r=5}^{n} 3r - 1 \text{ can be simplified to give } \frac{n-4}{2} [14 + (3n-1)] = \frac{n-4}{2} (13 + 3n)
\]

In such an instance, the lower bound **DOES NOT** need to start from 1.

**GP:** \[
\sum_{r=a}^{b} k^{g(r)} \text{ where } k \text{ is a real number constant and } g(r) \text{ must be a linear function in } r.
\]

\[
\sum_{r=3}^{2n} 2^{2r+1} = 2^7 + 2^9 + 2^{11} + \ldots + 2^{4n+1} \text{ is a GP with first term } 2^7, \text{ common ratio of } 2^2 = 4 \text{ and a total of } 2n-3+1=2n-2 \text{ terms.}
\]

Hence, \[
\sum_{r=3}^{2n} 2^{2r+1} \text{ can be simplified to give } \frac{2^7 (4^{2n-2} - 1)}{4-1} = \frac{128}{3} (4^{2n-2} - 1)
\]

In such an instance, the lower bound **DOES NOT** need to start from 1.
(c) Basic operations:

(i) \[ \sum_{r=a}^{b} f(r) \pm g(r) = \sum_{r=a}^{b} f(r) \pm \sum_{r=a}^{b} g(r) \]

(ii) \[ \sum_{r=a}^{b} kf(r) = k \sum_{r=a}^{b} f(r) \] where \( k \) is a real constant.

(iii) \[ \sum_{r=a}^{b} f(r) + \sum_{r=b+1}^{2b} f(r) = \sum_{r=a}^{2b} f(r) \quad \text{or} \quad \sum_{r=a}^{b} f(r) = \sum_{r=a}^{2b} f(r) - \sum_{r=b+1}^{2b} f(r) \]

(iv) \[ \sum_{r=a}^{b} h(k) = (b - a + 1)[h(k)] \] (Note that every term of the series is simply \( h(k) \)).

(v) If \( f(a) = 0 \) and \( f(a+1), f(a+2), f(a+3), \ldots, f(b) \) are all non-zero terms,

Then \[ \sum_{r=a}^{b} f(r) = \sum_{r=a+1}^{b} f(r) \]

2. Method of Differences

When partial fractions are typically involved along with the employment of the sigma notation, there is a very high likelihood MOD must be a main part of the solving strategy.

A mass cancellation shall be effected and the number of (surviving) terms normally collected would be an even quantity; of course this would depend on the number of linear factors constituting the denominator of the original block to be broken up by the partial fractions method.

Example: Express \( \frac{r}{(r+1)(r+2)(r+3)} \) in partial fractions, and hence show

that \[ \sum_{r=1}^{n} \frac{r}{(r+1)(r+2)(r+3)} = \frac{1}{4} + \frac{1}{2(n+2)} - \frac{3}{2(n+3)} \]

SOLUTIONS:

\[ \frac{r}{(r+1)(r+2)(r+3)} = \left( -\frac{1}{2} \right) \frac{1}{r+1} + 2 \frac{1}{r+2} + \left( -\frac{3}{2} \right) \frac{1}{r+3} \]

\[ = \frac{1}{2} \left[ -\frac{1}{r+1} + \frac{4}{r+2} - \frac{3}{r+3} \right] \] (shown)
3. Mathematical Induction

(a) Basic full structure outline:

Let $P_n$ be the proposition that .........................

For $P_1$: $LHS = .........$ $RHS = .........$

Since $LHS = RHS$, $P_1$ is true.

(Note: at times we may start from $P_2$ or $P_0$)
Assume $P_k$ is true for some $k \in \mathbb{Z}^+$, ie ………………….

Looking at $P_{k+1}$ : …………………………………………………… (THIS IS THE MAIN BODY, AND THE ABOVE ASSUMPTION MUST ALWAYS BE SUBSTITUTED INTO THIS PART IN ONE WAY OR ANOTHER)

Conclusion: $P_k$ is true $\Rightarrow P_{k+1}$ is true. Since $P_1$ is true, by Mathematical Induction, $P_n$ is true for all $n \in \mathbb{Z}^+$. (Note that if we start from $P_2$, then we have to revise the truth criteria to $n \geq 2$; if we start from $P_0$, we have to revise this to $n \geq 0$)

(b) Typical types of MI question structures:

(i) Testing the validity of a formula describing the summation of a series:

Example: Use induction to prove that $\sum_{r=2}^{n} (r^2 + r + 1)r! = (n + 1)^2 n! - 4$.

**SOLUTION FOR THE MAIN BODY:**

Assume $P_k$ is true, ie $\sum_{r=2}^{k} (r^2 + r + 1)r! = (k + 1)^2 k! - 4$

For $P_{k+1}$ : $\sum_{r=2}^{k+1} (r^2 + r + 1)r! = (k + 1)^2 k! - 4 + [(k + 1)^2 + (k + 1) + 1](k + 1)!$

$= (k + 1)[(k + 1)k!] + [(k + 1)^2 + (k + 1) + 1](k + 1)! - 4$

$= (k + 1)(k + 1)! + [(k + 1)^2 + (k + 1) + 1](k + 1)! - 4$

$= (k + 1)![(k + 1)^2 + 4k + 4] - 4$

$= (k + 1)![(k + 2)^2 - 4] = (k + 1)![(k + 1)^2 - 4] - 4$ (shown)

(ii) Testing the validity of a formula describing the generic term within a series:

Example: A sequence $u_0, u_1, u_2, \ldots$ is defined by $u_0 = -3$ and $u_{n+1} = 2u_n + 3^n + 5n$ for $n \geq 0$. Prove by mathematical induction that for all $n \geq 0$, $u_n = 2^n + 3^n - 5n - 5$. 

SOLUTION FOR THE MAIN BODY:

Assume \( P_k \) is true, ie \( u_k = 2^k + 3^k - 5k - 5 \)

For \( P_{k+1} \): \( u_{k+1} = 2u_k + 3^k + 5k \)

\[
= 2\left(2^k + 3^k - 5k - 5\right) + 3^k + 5k
\]

\[
= 2^{k+1} + 2\left(3^k\right) - 10k - 10 + 3^k + 5k
\]

\[
= 2^{k+1} + (2 + 1)\left(3^k\right) - 5k - 10
\]

\[
= 2^{k+1} + (3\left(3^k\right)) - (5k - 5) - 5
\]

\[
= 2^{k+1} + 3^{k+1} - 5(k + 1) - 5 \quad \text{(shown)}
\]

(c) Atypical question structure types:

Unexpected nasty problems could surface, in such cases we must be astute and sufficiently competent to work out the proof.

Example: Using the formula for \( \sin(A \pm B) \), prove that

\[
\sin\left(r + \frac{1}{2}\right) \theta - \sin\left(r - \frac{1}{2}\right) \theta = 2 \cos r \theta \sin \frac{1}{2} \theta.
\]

Hence find a formula for \( \sum_{r=1}^{n} \cos r \theta \) in terms of \( \sin \left(n + \frac{1}{2}\right) \theta \) and \( \sin \frac{1}{2} \theta \).

Prove by the method of mathematical induction that

\[
\sum_{r=1}^{n} \sin r \theta = \frac{\cos \frac{1}{2} \theta - \cos \left(n + \frac{1}{2}\right) \theta}{2 \sin \frac{1}{2} \theta}
\]

for all positive integers \( n \).

FULL SOLUTIONS:

\[
\sin\left(r + \frac{1}{2}\right) \theta - \sin\left(r - \frac{1}{2}\right) \theta = \sin\left(r \theta + \frac{1}{2} \theta\right) - \sin\left(r \theta - \frac{1}{2} \theta\right)
\]
\[
= \sin r\theta \cos \frac{1}{2}\theta + \cos r\theta \sin \frac{1}{2}\theta - \left( \sin r\theta \cos \frac{1}{2}\theta - \cos r\theta \sin \frac{1}{2}\theta \right)
\]

\[
= 2 \cos r\theta \sin \frac{1}{2}\theta \quad \text{(shown)}
\]

\[
\sum_{r=1}^{n} \left[ \sin \left( r + \frac{1}{2} \right)\theta - \sin \left( r - \frac{1}{2} \right)\theta \right] = \sum_{r=1}^{n} 2 \cos r\theta \sin \frac{1}{2}\theta
\]

\[
\sum_{r=1}^{n} \cos r\theta = \frac{1}{2 \sin \frac{1}{2}\theta} \sum_{r=1}^{n} \left[ \sin \left( r + \frac{1}{2} \right)\theta - \sin \left( r - \frac{1}{2} \right)\theta \right] \quad \text{(note that } 2 \sin \frac{1}{2}\theta \text{ is a constant and can be isolated outside the sigma notation.)}
\]

\[
= \frac{1}{2 \sin \frac{1}{2}\theta} \left[ \sin \frac{3}{2}\theta - \sin \frac{1}{2}\theta \right] = \frac{1}{2 \sin \frac{1}{2}\theta} \left[ \sin \left( n + \frac{1}{2} \right)\theta - \sin \frac{1}{2}\theta \right] \quad \text{(shown)}
\]

Let \( P_n \) be the hypothesis that \( \sum_{r=1}^{n} \sin r\theta = \frac{\cos \frac{1}{2}\theta - \cos \left( n + \frac{1}{2} \right)\theta}{2 \sin \frac{1}{2}\theta}, \quad n \in Z^+ \).
For $P_1$: $LHS = \sin \theta$; $RHS = \frac{\cos \frac{1}{2} \theta - \cos \left( \frac{3}{2} \theta \right)}{2 \sin \frac{1}{2} \theta} = \frac{\cos \left( \frac{3}{2} \theta \right) - \cos \frac{1}{2} \theta}{-2 \sin \frac{1}{2} \theta} = \frac{-2 \sin \theta \sin \frac{1}{2} \theta}{-2 \sin \frac{1}{2} \theta} = \sin \theta$

[$\because \cos A - \cos B = -2 \sin \left( \frac{A + B}{2} \right) \sin \left( \frac{A - B}{2} \right)$ is used above]

$LHS = RHS$, $\therefore P_1$ is true.

Assume $P_k$ is true form some $k \in \mathbb{Z}^+$, ie $\sum_{r=1}^{k} \sin r \theta = \frac{\cos \frac{1}{2} \theta - \cos \left( k + \frac{1}{2} \theta \right)}{2 \sin \frac{1}{2} \theta}$

For $P_{k+1}$: $\sum_{r=1}^{k+1} \sin r \theta = \frac{\cos \frac{1}{2} \theta - \cos \left( k + \frac{1}{2} \theta \right)}{2 \sin \frac{1}{2} \theta} + \sin(k + 1) \theta$

$= \frac{\cos \frac{1}{2} \theta - \cos \left( k + \frac{1}{2} \theta \right) + 2 \sin(k + 1) \theta \sin \frac{1}{2} \theta}{2 \sin \frac{1}{2} \theta}$

Using a variation of the formula $\cos A - \cos B = -2 \sin \left( \frac{A + B}{2} \right) \sin \left( \frac{A - B}{2} \right)$

ie $2 \sin \left( \frac{A + B}{2} \right) \sin \left( \frac{B - A}{2} \right) = \cos A - \cos B$, where $A = \left( k + \frac{1}{2} \right) \theta$ and $B = \left( k + \frac{3}{2} \right) \theta$. 
\[
\cos \frac{1}{2} \theta - \cos \left(k + \frac{1}{2}\right) \theta + \cos \left(k + \frac{1}{2}\right) \theta - \cos \left(k + \frac{3}{2}\right) \theta
\]

(1) becomes:

\[
\frac{\cos \frac{1}{2} \theta - \cos \left(k + \frac{3}{2}\right) \theta}{2 \sin \frac{1}{2} \theta} = \frac{\cos \frac{1}{2} \theta - \cos \left[(k + 1) + \frac{1}{2}\right] \theta}{2 \sin \frac{1}{2} \theta}
\]

\[P_k \text{ is true} \implies P_{k+1} \text{ is true; since } P_1 \text{ is true, therefore by mathematical induction,}
\]

\[
\sum_{r=1}^{n} \sin r \theta = \frac{\cos \frac{1}{2} \theta - \cos \left(n + \frac{1}{2}\right) \theta}{2 \sin \frac{1}{2} \theta}
\]

is true for all positive integers \(n\). (shown)